ON THE VOLUME OF LATTICE POLYHEDRA

By J. E. REEVE

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Introduction

LET l denote the fundamental lattice in the euclidean plane E_2 consisting of all points with integral coordinates in some preassigned cartesian coordinate system of E_2. A non-empty subset γ of E_2 will be called an *l-polygon* if, firstly, it admits a finite simplicial covering† by rectilinear simplexes whose vertices belong to l and, secondly, it is pure of dimension two. It follows in particular that l-polygons are closed subsets of the plane. It is well known‡ that the area $A(\gamma)$ of an l-polygon γ whose boundary is a Jordan curve $\bar{\gamma}$ is given by the formula

$$A(\gamma) = l(\gamma) - \tfrac{1}{2}l(\bar{\gamma}) - 1, \tag{1}$$

where $l(\gamma)$ and $l(\bar{\gamma})$ denote the number of points of l which belong respectively to γ and its boundary $\bar{\gamma}$, provided the fundamental parallelogram is of unit area.

In this note we discuss certain generalizations of (1) and, in particular, we obtain in Theorem II a formula, which is in many ways analogous to the one above, for the volume of a polyhedron of a particular type in three-dimensional euclidean space E_3. The class of polyhedra for which our formula is valid includes as a special case the class of convex polyhedra whose vertices have integral coordinates in some cartesian coordinate system in E_3. The formula for convex polyhedra is stated explicitly in Theorem I. That this formula must embody something more than the direct extension of (1) which one might at first anticipate is clearly illustrated by the following example.

Consider the tetrahedron τ whose vertices, in some cartesian coordinate system of E_3, are the points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(1,1,r)$; r being a positive integer. The four vertices of τ belong to the lattice consisting of all points with integral coordinates and furthermore, as is easily verified, this lattice contains no other points of τ. Thus τ is one of the simplest convex polyhedra of the type we wish to consider and it has the somewhat

† An account of the idea of a simplicial covering may be found, for example, in *Lehrbuch der Topologie*, by H. Seifert and W. Threlfall (Chelsea Publishing Company, 1947).

‡ For a proof of this in the case when $A = \frac{1}{2}$ see Hardy and Wright, *The Theory of Numbers*, 3rd edn. (Oxford, 1954), chapter iii, 'A Theorem of Minkowski'.

disturbing property that, by suitable choice of the integer r, its volume may be made as large as we please without altering the numbers of lattice points lying on its boundary and in its interior.

The way in which we get over this difficulty is to bring into consideration not only the basic lattice of points with integral coordinates but also a secondary lattice which, in the simplest case, consists of all points each of whose coordinates is a multiple of $\frac{1}{2}$. We feel that the interest of our results lies in some respects not so much in the form of the formula we obtain for the volume of a polyhedron as in the fact that the introduction of a secondary lattice enables us to obtain such a formula at all. We return to this question in the concluding section of this paper, and we believe that we do obtain there not only a partial explanation of the existence of our formula for the volume of polyhedra in three-dimensional space but also some indication of what might be expected in an analogous investigation in space of dimension greater than three.

An interesting fact which emerged, as we shall explain later, from our initial attempts to obtain a formula for the volume of a polyhedron in three dimensions is that in some cases, including in particular the case of a convex polyhedron whose vertices are lattice points, there exists a certain identical relation connecting the numbers of points which the boundary of a given polyhedron has in common with each of the two lattices introduced above. This is in some way a reflection of the fact that in the case of these special polyhedra the secondary lattice provides, as it were, a certain amount of redundant information. This question is clarified in the footnote to Theorem III.

We would mention finally, not so much to put it on record for the sake of its own interest as to give an indication of the direction in which we turn for our generalization in three dimensions, that formula (1) admits an extension whose range of validity covers all l-polygons in E_2 and not merely those whose boundaries are Jordan curves. In fact, *if γ is an arbitrary l-polygon of area $A(\gamma)$ and $\bar{\gamma}$ denotes its boundary, i.e. set of boundary points, and if $l(\gamma)$ and $l(\bar{\gamma})$ denote the number of points of l lying respectively on γ and $\bar{\gamma}$, then*

$$A(\gamma) = \{l(\gamma)+N(\gamma)\}-\tfrac{1}{2}\{l(\bar{\gamma})+N(\bar{\gamma})\}, \tag{2}$$

where $N(\gamma)$ and $N(\bar{\gamma})$ denote the respective Euler–Poincaré characteristics of γ and $\bar{\gamma}$. Here and throughout this paper the Euler–Poincaré characteristic of any n-dimensional complex admitting a finite simplicial covering containing α_ν simplexes of dimension ν, $0 \leqslant \nu \leqslant n$, is taken to be the integer $\sum_{\nu=0}^{n} (-1)^{\nu+1}\alpha_\nu$. A proof that the Euler–Poincaré characteristic is not dependent upon the particular simplicial covering chosen may be found, for example, in *Lehrbuch der Topologie* to which reference has already been made in a footnote.

The validity of (2) may be deduced from that of (1) without difficulty by regarding the given l-polygon as a union of l-polygons the boundary of each of which is a Jordan curve, and then proving that the expression on the right of (2) enjoys an additive property with respect to the union of any two l-polygons having an intersection of dimension at most one. It hardly seems necessary to give a detailed proof of this result here, and although in what follows we shall have occasion to make use of (1) we shall not require the more general formula (2). However, it may be worth remarking that if P is any point of the plane and we define $m(\gamma, P)$ by

$$m(\gamma, P) = \lim_{\epsilon \to 0} \frac{A(\gamma, P, \epsilon)}{\pi\epsilon^2},$$

where $A(\gamma, P, \epsilon)$ denotes the area of the region of the plane common to both γ and a circle of radius ϵ and centre P, then (1) and (2) can each be written in the form

$$A(\gamma) = \sum_{P \in l} m(\gamma, P). \tag{3}$$

For, when $\bar{\gamma}$ is a Jordan curve it is fairly evident that the right-hand sides of (1) and (3) are equal. Further, it is clear that each side of (3) is additive with respect to the union of any two l-polygons having an intersection of dimension at most one, and this makes it easy to prove the equivalence of (2) and (3) for any l-polygon and hence to provide a simple method of establishing (1).

I am indebted to the referee for a number of helpful suggestions and in particular for an alternative version of my original proof of the formula (10), basing it on his Lemma III.

1. Preliminary definitions

We fix, once and for all, in a euclidean space E_3, a system of cartesian coordinates such that the unit cell is of unit volume. This means that we choose three arbitrary linearly independent vectors such that the parallelepiped spanned by them is of unit volume and we define coordinates with respect to this triplet of vectors. The set of points with integral coordinates forms the *fundamental lattice* which throughout this paper will be denoted by L. For each positive integer n we define a further lattice L_n as follows. The point (a, b, c) belongs to L_n if and only if the point (na, nb, nc) belongs to L. We note that with this definition L_1 coincides with L.

A subset Γ of E_3 will be called a *singular polyhedron* if, whenever it is not empty, it admits a finite rectilinear simplicial covering, that is, a finite simplicial covering by rectilinear simplexes. The Euler–Poincaré characteristic of a singular polyhedron Γ will be denoted by $N(\Gamma)$ and its volume by $V(\Gamma)$. The symbol $L_n(\Gamma)$ will be used to denote the number of points of L_n which belong to Γ, and we define, for any singular polyhedron Γ and positive integer n, a function $M_n(\Gamma)$ as follows,

$$M_n(\Gamma) = L_n(\Gamma) - nL(\Gamma) - (n-1)N(\Gamma).$$

We now define some special types of singular polyhedron.

The convex hull of a finite set of points in E_3 will be called a *convex polygon* provided this convex hull is of dimension two. The intersection of a convex polygon γ with one of its planes of support in E_3 is called respectively an *edge* or *vertex* of γ according as this intersection is of dimension one or zero. The convex hull of a finite set of points in E_3 will be called a *convex polyhedron* provided this convex hull is of dimension three. The intersection of a convex polyhedron Γ with one of its planes of support is called respectively a *face*, *edge*, or *vertex* of Γ according as this intersection is of dimension two, one, or zero. The boundary (i.e. set of boundary points) of a convex polyhedron Γ is the union of its faces and will be denoted by $\bar{\Gamma}$. If Γ is a convex polyhedron, the faces of Γ are convex polygons, each edge and vertex of Γ is an edge or vertex of at least one of the faces of Γ, and conversely, each edge and vertex of any face of Γ is also an edge or vertex of Γ itself. Lastly, it is well known that every convex polyhedron admits a finite rectilinear simplicial covering and so is a singular polyhedron in the sense defined above. Actually we shall shortly be proving a slightly stronger result than this. We now turn to some definitions of a somewhat different nature.

A subset Π of E_3 will be called an *L-polyhedron* if:

(i) Π is a non-empty singular polyhedron,

(ii) Π is pure of dimension three, and

(iii) Π admits a rectilinear simplicial covering all of whose vertices belong to the lattice L.

The *boundary* $\bar{\Pi}$ of an L-polyhedron Π is the set of boundary points of Π, in the set-theoretical sense. As a particular case of an L-polyhedron we have an *L-tetrahedron*; this is simply a 3-simplex each of whose vertices belong to L.†

A subset π of E_3 will be called a *singular L-surface* if:

(i) π is a singular polyhedron,

(ii) π is of dimension two at most, and

(iii) π, if not empty, admits a rectilinear simplicial covering all of whose vertices belong to L.

† Here, as elsewhere in this paper, simplexes are understood to be closed.

A subset π of E_3 will be called an *unbranched L-surface* if:

(i) π is a singular L-surface,

(ii) π is non-empty and pure of dimension two, and

(iii) π admits a rectilinear simplicial covering K whose vertices belong to L and which has the additional property that none of its 1-simplexes are incident with more than two of its 2-simplexes.

The *boundary* of the unbranched L-surface π will be denoted by $\bar{\pi}$ and is defined as the union of all the 1-simplexes of the covering K which are incident with only one 2-simplex of K.† As a particular case of an unbranched L-surface we have an *L-triangle*; this is simply a 2-simplex whose three vertices belong to L.

A subset p of E_3 will be called a *singular L-path* if:

(i) p is a singular polyhedron,

(ii) p is of dimension at most one, and

(iii) p, if not empty, admits a rectilinear simplicial covering all of whose vertices belong to L.

As a special case of a singular L-path we have an *L-segment*; this is just a 1-simplex whose end-points belong to L.

2. Statement of the theorems

With the definitions and notations introduced in § 1 we can now state the three following theorems.

THEOREM I. *Let n be an integer greater than unity and let Γ be any convex polyhedron all of whose vertices belong to the lattice L. Then*

$$2(n-1)n(n+1)V(\Gamma) = 2\{L_n(\Gamma)-nL(\Gamma)\}-\{L_n(\bar{\Gamma})-nL(\bar{\Gamma})\}, \tag{4}$$

and, in addition, we have the following relation

$$L_n(\bar{\Gamma})-n^2L(\bar{\Gamma}) = 2(1-n^2). \tag{5}$$

The hypotheses of this theorem include the condition that n be greater than unity, but we note that both (4) and (5) are trivially satisfied if we put $n = 1$.

As we shall see later, the relations (4) and (5) are special cases of more general ones of which the two following theorems give an explicit account.

THEOREM II. *Let n be a positive integer and let Π be any L-polyhedron. Then*

$$2(n-1)n(n+1)V(\Pi) = 2M_n(\Pi)-M_n(\bar{\Pi}). \tag{6}$$

† It would seem to be out of place to give here a proof of the fact that 'boundary' in this sense does in fact not depend upon the choice of the covering K.

THEOREM III. *Let n be a positive integer. If π is any unbranched L-surface then*†

$$2L_n(\pi)-2n^2L(\pi)+2(1-n^2)N(\pi)+nL_n(\bar{\pi})-nL(\bar{\pi}) = 0. \tag{7}$$

These theorems are trivially verified if $n = 1$. However, if $n > 1$ the formula (6) would enable us to calculate the volume of Π.

Before we prove these theorems it may perhaps be of interest to mention that the formulae of Theorem I were originally obtained in the case when $n = 2$ by assuming that the volume of a convex polyhedron whose vertices belonged to L could be expressed as a linear combination of seventeen terms which arose as follows. The points of L_2 can be divided into four classes depending upon the number of their three coordinates which are integers, the points of these classes may again be divided into four groups according as to whether they lie in the interior of the given convex polyhedron, in the interior of one of its faces, in the interior of one of its edges, or finally, at one of its vertices. The seventeenth term was an additional constant. Three of these terms can, of course, be discounted at once in view of the fact that the vertices of the polyhedron belong to L and so have all their coordinates integral. The coefficients of the remaining terms were found by substituting values found for various simple convex polyhedra and solving the resulting simultaneous linear equations. The formula for the volume which was obtained in this way was equivalent to

$$2(n-1)n(n+1)V(\Gamma) = 2\{L_n(\Gamma)-nL(\Gamma)\}-$$
$$-\{L_n(\bar{\Gamma})-nL(\bar{\Gamma})\}+\lambda\{L_n(\bar{\Gamma})-n^2L(\bar{\Gamma})-2(1-n^2)\},$$

where λ appeared to be an arbitrary parameter. This is, of course, quite in accordance with the assertions contained in Theorem I, but we mention it for what it is worth because it shows how we were led to find the relation (5) in addition to the formula (4) for which we were looking.

3. Proof of Theorem I

In this section we show that Theorem I is in fact a consequence of Theorems II and III.

In the first place we notice that if Γ is a convex polyhedron whose vertices belong to L then $N(\Gamma) = -1$, $N(\bar{\Gamma}) = -2$, and the boundary of $\bar{\Gamma}$ is the empty set, $\bar{\Gamma}$ itself being, as we shall see in a moment, an unbranched L-surface. It is now easily verified that if we replace Π by Γ in formula (6) the latter reduces to (4), and if we replace π by $\bar{\Gamma}$ in (7) then the latter

† We would point out that if an L-polyhedron Π has a boundary $\bar{\Pi}$ which is an unbranched L-surface then Theorem III is applicable to $\bar{\Pi}$ and furnishes between Π and L a relation to which we have already made reference.

reduces to (5). Hence, in order to deduce Theorem I from Theorems II and III, it will be sufficient to show that if Γ is a convex polyhedron all of whose vertices belong to L then Γ is an L-polyhedron and its boundary $\bar{\Gamma}$ is an unbranched L-surface. We shall in fact prove the slightly stronger

LEMMA I. *If Γ is a convex polyhedron all of whose vertices belong to the lattice L then Γ admits at least one rectilinear simplicial covering the set of whose vertices coincides with the set of vertices of Γ.*

This lemma certainly ensures that Γ is an L-polyhedron and, if we remember that the boundary of a convex polyhedron is a homeomorphic image of a 2-sphere, it also implies that $\bar{\Gamma}$ is an unbranched L-surface.

Proof of Lemma I. Let γ be any face of the convex polyhedron Γ. Since all the vertices of Γ belong to L it follows that all the vertices of the convex polygon γ also belong to L. Thus if we join one vertex of γ to all the remaining vertices and edges of γ with which it is not incident we obtain a triangulation of γ the vertices of which coincide with the vertices of γ. We have already mentioned that each edge and vertex of Γ is an edge or vertex of at least one face of Γ and conversely, each edge and vertex of any face of Γ is also an edge or vertex of Γ itself. It therefore follows that if we triangulate each face of Γ in the way just described then we obtain a simplicial covering of the boundary $\bar{\Gamma}$ of Γ and the set of vertices of this covering coincides with the set of vertices of Γ. Let P be any vertex of Γ, and let K be a covering of $\bar{\Gamma}$, constructed as above, but in such a way that the triangulation of any face of Γ incident with P is formed by joining vertices and edges of that face to P. A simplicial covering of Γ of the type required by the lemma can now be obtained by joining P to each simplex of K with which it is not incident. This completes the proof of Lemma I.

We have thus seen that Theorem I is a consequence of Theorems II and III and it now only remains to establish these two latter theorems. We devote the next section to the proof of Theorem III.

4. Proof of Theorem III

A preliminary lemma

We shall require both in this and a subsequent section the following lemma.

LEMMA II. *If n be a positive integer and p a singular L-path then* $M_n(p) = 0$.

Proof of Lemma II. Suppose p_1 and p_2 to be two singular L-paths whose intersection p^*, if not vacuous, consists of a finite set of points belonging to L. The point set $p_1 \cup p_2$ is again a singular L-path which we shall denote by p_0. It is easily verified that

$$L_n(p_0) = L_n(p_1)+L_n(p_2)-L_n(p^*),$$
$$L(p_0) = L(p_1)+L(p_2)-L(p^*),$$
and
$$N(p_0) = N(p_1)+N(p_2)-N(p^*).$$
Hence
$$M_n(p_0) = M_n(p_1)+M_n(p_2)-M_n(p^*).$$

Now, by definition,

$$M_n(p^*) = L_n(p^*)-nL(p^*)-(n-1)N(p^*),$$

and since p^* is a discrete set of, say, k points $(0 \leqslant k < \infty)$ belonging to L, which means that $L_n(p^*) = L(p^*) = -N(p^*) = k$, it follows that

$$M_n(p^*) = 0.$$

Thus
$$M_n(p_0) = M_n(p_1)+M_n(p_2).$$

By repeated application of this last result we can reduce the proof of the lemma to a trivial verification that the function M_n vanishes for the special singular L-paths consisting either of the empty set, of a single point of L, or of a single L-segment.

The additive property of the function $G_n(\pi)$

If n is a positive integer and π is an unbranched L-surface we define the function $G_n(\pi)$ as follows

$$G_n(\pi) = 2L_n(\pi)-2n^2L(\pi)+2(1-n^2)N(\pi)+nL_n(\bar{\pi})-nL(\bar{\pi}).$$

Theorem III then states that $G_n(\pi) = 0$. As a first step towards proving this theorem we shall show that G_n has the following additive property.

Let an unbranched L-surface π_0 be the union of two unbranched L-surfaces π_1 and π_2 whose intersection is a singular L-path lying on each of $\bar{\pi}_1$ and $\bar{\pi}_2$, the boundaries of π_1 and π_2 respectively. The function G_n, n being a positive integer, then enjoys the property that

$$G_n(\pi_0) = G_n(\pi_1)+G_n(\pi_2).$$

Proof. The intersection $\pi_1 \cap \pi_2$ is a, possibly vacuous, singular L-path which we shall denote by p. Define $p^* = \bar{\pi}_0 \cap p$, where, in accordance with the notation introduced earlier, $\bar{\pi}_0$ is the boundary of π_0.

It is easily verified that

$$L_n(\pi_0) = L_n(\pi_1)+L_n(\pi_2)-L_n(p),$$
$$L(\pi_0) = L(\pi_1)+L(\pi_2)-L(p),$$
$$N(\pi_0) = N(\pi_1)+N(\pi_2)-N(p),$$
$$L_n(\bar{\pi}_0) = L_n(\bar{\pi}_1)+L_n(\bar{\pi}_2)-2L_n(p)+L_n(p^*),$$
and
$$L(\bar{\pi}_0) = L(\bar{\pi}_1)+L(\bar{\pi}_2)-2L(p)+L(p^*).$$
Hence
$$G_n(\pi_0) = G_n(\pi_1)+G_n(\pi_2)-H_n(p,p^*),$$

where

$$H_n(p,p^*) = 2(n+1)L_n(p) - \\ -2n(n+1)L(p) - 2(n^2-1)N(p) - nL_n(p^*) + nL(p^*).$$

Now the point set p^*, if not empty, consists† of a finite number of points belonging to L and so $L_n(p^*) = L(p^*)$. From this it follows that

$$H_n(p,p^*) = 2(n+1)M_n(p),$$

and so in virtue of Lemma II we can conclude that $H_n(p,p^*) = 0$ and hence that

$$G_n(\pi_0) = G_n(\pi_1) + G_n(\pi_2).$$

Reduction to fundamental L-triangles

We may use the additive property that we have just proved to replace Theorem III by an equivalent but much simpler statement concerning L-triangles. In fact, we know that an unbranched L-surface π admits a finite simplicial covering K whose vertices belong to L. Let σ be one of the 2-simplexes of K and suppose that $\pi - \sigma$ is not empty. σ is then an L-triangle and the closure, π_1, of the set $\pi - \sigma$ is an unbranched L-surface. Further, π, π_1, and σ satisfy the hypotheses made on π_0, π_1, and π_2 respectively in the statement of the additive property of G_n in the previous paragraph. The 2-simplexes of K may be removed one by one in this manner and thus, since K is finite, we see that it will only be necessary, in order to establish Theorem III, to prove that $G_n(\sigma) = 0$ for an arbitrary L-triangle σ. We can in fact do a little better than this if we introduce what may be called a *fundamental L-triangle*, that is, an L-triangle which contains no points of L other than its three vertices. For, if σ is an arbitrary L-triangle, it will contain at most a finite number of points of L and if it is not fundamental it will contain at least one point, say P, of L distinct from any of its three vertices. By joining P to each of the vertices and edges of σ with which it is not incident we obtain a simplicial covering of σ each of the 2-simplexes of which contain fewer points of L than σ does. Continuing the subdivision in this way leads to a simplicial covering of σ in which every 2-simplex is a fundamental L-triangle. By using the additive property of G_n and the same arguments as above we arrive at the conclusion that in order to prove Theorem III it will be sufficient to verify that $G_n(\sigma) = 0$ for an arbitrary fundamental L-triangle σ.

† Any point P of p which does not belong to L must belong, in virtue of our definitions, to both the interior of an edge of some 2-simplex σ_1 of some simplicial covering of π_1, and the interior of an edge of some 2-simplex σ_2 of some simplicial covering of π_2. Since p is a singular L-path and P does not belong to L it follows that σ_1 and σ_2 have a one-dimensional intersection and so P must be an interior point of π_0.

Proof for fundamental L-triangles

We suppose that (x, y, z) are the current coordinates of a point of E_3 in the preassigned cartesian coordinate system which determines the lattice L. As usual, n is a fixed positive integer. If σ is an arbitrary fundamental L-triangle of E_3 then the numbers $L_n(\sigma)$, $L(\sigma)$, $N(\sigma)$, $L_n(\bar{\sigma})$, $L(\bar{\sigma})$ and hence also $G_n(\sigma)$ remain invariant under unimodular transformations of E_3. We can therefore without loss of generality restrict our attention to a fundamental L-triangle σ_0 lying in the plane $z = 0$ and having as vertices the three points $(0,0,0)$, $(p,0,0)$, and $(q,r,0)$; p, q, and r being integers satisfying the inequalities $0 < p$ and $0 \leqslant q < r$. Now since σ_0 is a fundamental L-triangle we see at once that $p = 1$, for otherwise the edge of σ_0 joining the vertices $(0,0,0)$ and $(p,0,0)$ would contain points of L other than these two vertices. In addition, again from the fact that σ_0 is fundamental, we can deduce with the aid of formula (1) that the area of σ_0 is $\frac{1}{2}$. This implies that $r = 1$ and hence that $q = 0$. Thus the vertices of σ_0 must be the points $(0,0,0)$, $(1,0,0)$, and $(0,1,0)$. It is now easily verified that

$$L(\sigma_0) = 3, \qquad L_n(\sigma_0) = \tfrac{1}{2}(n+1)(n+2),$$
$$L(\bar{\sigma}_0) = 3, \qquad L_n(\bar{\sigma}_0) = 3n, \qquad N(\sigma_0) = -1,$$

and hence that $G_n(\sigma_0) = 0$. This completes the proof of Theorem III.

5. Proof of Theorem II

Although the details of the proof of Theorem II are somewhat more involved than those of the preceding proofs of Lemma II and Theorem III the same general pattern can be observed. That is to say, we first establish an additive property and then use this to reduce the proof of the theorem to a verification for a particularly simple polyhedron. We start by proving the additive property.

The additive property of $\Omega_n(\Pi)$

Given an L-polyhedron Π and positive integer n we define functions $W_n(\Pi)$ and $\Omega_n(\Pi)$ as follows

$$W_n(\Pi) = 2M_n(\Pi) - M_n(\bar{\Pi}),$$

and
$$\Omega_n(\Pi) = 2(n-1)n(n+1)V(\Pi) - W_n(\Pi).$$

Theorem II then asserts that $\Omega_n(\Pi) = 0$.

The first thing we shall show is that $\Omega_n(\Pi)$ enjoys the following additive property.

If an L-polyhedron Π_0 is the union, $\Pi_1 \cup \Pi_2$, of two L-polyhedra Π_1 and Π_2 whose, possibly vacuous, intersection is a singular L-surface, then

$$\Omega_n(\Pi_0) = \Omega_n(\Pi_1) + \Omega_n(\Pi_2).$$

Although they are in fact contained implicitly in the foregoing hypotheses, for the sake of simplicity, we prefer to adjoin to the above the following additional assumptions. Firstly, we assume that the singular L-surface $\pi = \Pi_1 \cap \Pi_2$ lies on each of the boundaries $\bar{\Pi}_1$ and $\bar{\Pi}_2$ of Π_1 and Π_2 respectively, and, secondly, we assume that $\pi^* = \bar{\Pi}_0 \cap \pi = \bar{\Pi}_0 \cap \bar{\Pi}_1 \cap \bar{\Pi}_2$ is a singular L-path, $\bar{\Pi}_0$ being, as in our usual notation, the boundary of Π_0.

To prove the additive property of Ω_n we note first of all that the above hypotheses enable us to verify without difficulty that

$$L_n(\Pi_0) = L_n(\Pi_1)+L_n(\Pi_2)-L_n(\pi),$$
$$L(\Pi_0) = L(\Pi_1)+L(\Pi_2)-L(\pi),$$
$$N(\Pi_0) = N(\Pi_1)+N(\Pi_2)-N(\pi),$$
$$L_n(\bar{\Pi}_0) = L_n(\bar{\Pi}_1)+L_n(\bar{\Pi}_2)-2L_n(\pi)+L_n(\pi^*),$$
$$L(\bar{\Pi}_0) = L(\bar{\Pi}_1)+L(\bar{\Pi}_2)-2L(\pi)+L(\pi^*),$$

and
$$N(\bar{\Pi}_0) = N(\bar{\Pi}_1)+N(\bar{\Pi}_2)-2N(\pi)+N(\pi^*).$$

Thus
$$M_n(\Pi_0) = M_n(\Pi_1)+M_n(\Pi_2)-M_n(\pi)$$

and
$$M_n(\bar{\Pi}_0) = M_n(\bar{\Pi}_1)+M_n(\bar{\Pi}_2)-2M_n(\pi)+M_n(\pi^*)$$

and hence
$$W_n(\Pi_0) = W_n(\Pi_1)+W_n(\Pi_2)-M_n(\pi^*),$$

but in virtue of Lemma II $M_n(\pi^*)$ vanishes and thus

$$W_n(\Pi_0) = W_n(\Pi_1)+W_n(\Pi_2).$$

The additive property of Ω_n now follows at once on account of the fact that

$$V(\Pi_0) = V(\Pi_1)+V(\Pi_2).$$

Reduction to fundamental tetrahedra

If Π is an arbitrary L-polyhedron then we know that Π admits a finite rectilinear simplicial covering K whose vertices belong to L. Let τ be one of the 3-simplexes of K. If $\Pi-\tau$ is not empty then Π, τ, and the closure of $\Pi-\tau$ are L-polyhedra satisfying the hypotheses made on Π_0, Π_1, and Π_2 respectively in the statement of the additive property of Ω_n proved in the previous paragraph. Furthermore, this process of removing 3-simplexes from the finite simplicial complex K may be repeated until all the 3-simplexes have been removed, and at each stage the hypotheses ensuring the additivity of Ω_n are satisfied. Thus in order to establish Theorem II we have only to verify that $\Omega_n(\tau)$ vanishes for an arbitrary L-tetrahedron τ. If we define a *fundamental L-tetrahedron* to be an L-tetrahedron which contains no points of L other than its four vertices then we can reduce the proof of Theorem II still further. In fact, if τ is an arbitrary L-tetrahedron it will contain at most a finite number of points of L and if it is not a fundamental L-tetrahedron it will contain at least one point, say P, of L distinct from any of its vertices. By joining P to each of the faces, edges, and vertices of τ with which it is not incident we obtain a simplicial covering of τ each of the 3-simplexes of which contains fewer points of L than τ does. By continuing to subdivide the simplexes in this way we can obtain a finite simplicial covering of τ in which every 3-simplex is a fundamental L-tetrahedron. By using the additive property of Ω_n and the same arguments as already employed above we can conclude that in order to prove Theorem II it will be sufficient to verify that $\Omega_n(\tau) = 0$ for an arbitrary fundamental L-tetrahedron τ.

Proof for fundamental L-tetrahedra

If τ is an arbitrary fundamental L-tetrahedron and n a fixed positive integer we wish, in order to complete the proof of Theorem II, to show that $\Omega_n(\tau) = 0$, or more fully, that

$$2(n-1)n(n+1)V(\tau) = 2\{L_n(\tau)-nL(\tau)-(n-1)N(\tau)\} \\ -\{L_n(\bar{\tau})-nL(\bar{\tau})-(n-1)N(\bar{\tau})\}. \tag{8}$$

Now since τ is a fundamental L-tetrahedron we must have $L(\tau) = 4$, $L(\bar{\tau}) = 4$, $N(\tau) = -1$, $N(\bar{\tau}) = -2$ and if $\Psi_n(\tau)$ denotes the number of points of L_n which lie in the interior of τ then $L_n(\tau) = L_n(\bar{\tau})+\Psi_n(\tau)$. Finally, if σ is one of the faces of τ then since τ is a fundamental L-tetrahedron σ is a fundamental L-triangle. Thus, as we have seen in the previous section, $L_n(\sigma) = \frac{1}{2}(n+1)(n+2)$. It is also easy to verify that each edge of τ contains exactly $n+1$ points of L_n. From this it follows that

$$L_n(\bar{\tau}) = 4.\tfrac{1}{2}(n+1)(n+2)-6(n+1)+4 = 2(n^2+1).$$

We see, therefore, on substituting these values in (8), that we have to verify that

$$(n-1)n(n+1)V(\tau) = \Psi_n(\tau)+(n-1)^2. \tag{9}$$

Before looking into this last relation more closely we remark that under a unimodular transformation of E_3 τ will be transformed into another fundamental L-tetrahedron and that both $V(\tau)$ and $\Psi_n(\tau)$ will remain invariant. We can therefore make use of a unimodular transformation to simplify our problem.

We suppose that (x,y,z) are the current coordinates of a point of E_3 in the preassigned cartesian coordinate system which determines the lattice L, and we assume that one of the vertices of τ lies at the origin of these coordinates. There then exists a unimodular transformation which transforms τ into a fundamental L-tetrahedron τ_0 with vertices $P_0(0,0,0)$, $P_1(p_1,0,0)$, $P_2(p_2,q_2,0)$, and $P_3(p_3,q_3,r_3)$, where p_1, p_2, q_2, p_3, q_3, and r_3 are integers satisfying the inequalities $0 < p_1$, $0 \leqslant p_2 < q_2$, $0 \leqslant p_3 < r_3$, and

$0 \leqslant q_3 < r_3$. We can, however, be a little more explicit than this for, in the first place, since τ_0 is fundamental, each edge, and in particular the edge P_0P_1, of τ_0 can contain no points of L other than its end-points, thus $p_1 = 1$. Again, the L-triangle $P_0P_1P_2$ can contain no points of L other than its three vertices and so, in view of formula (1), its area must be $\frac{1}{2}$ and hence $q_2 = 1$ and $p_2 = 0$. It can be shown without difficulty that the fact that the remaining three faces of τ_0 are fundamental L-triangles leads to the conditions

$$(p_3, r_3) = 1, \quad (q_3, r_3) = 1, \quad \text{and} \quad (p_3+q_3-1, r_3) = 1,$$

where, if a and b are integers not both of which are zero, (a, b) denotes their greatest common factor. For example, to prove the last of these conditions we may write $(p_3+q_3-1, r_3) = d$ and $a = (p_3+q_3-1)/d$, $c = r_3/d$. Then, since

$$(a+1, 0, c) = \left(1-\frac{1}{d}+\frac{q_3}{d}\right)(1,0,0)-\frac{q_3}{d}(0,1,0)+\frac{1}{d}(p_3, q_3, r_3),$$

the triangles with vertices

$$(1,0,0),\ (0,1,0),\ (p_3, q_3, r_3),$$

and

$$(1,0,0),\ (0,1,0),\ (a+1, 0, c)$$

are coplanar L-triangles. So, if the first is to be a fundamental L-triangle, its area cannot exceed the area of the second L-triangle. But by projecting onto the plane $x = 0$, we see that the ratio of the areas of these two triangles is the ratio of the areas of the triangles with vertices

$$(0,0),\ (1,0),\ (q_3, r_3),$$

and

$$(0,0),\ (1,0),\ (0,c),$$

which is $r_3/c = d$. Hence $d = 1$ as required. Writing $p = p_3$, $q = q_3$, and $r = r_3$ we can summarize these results as follows.

There is a unimodular transformation of E_3 which carries the given fundamental L-tetrahedron τ into the fundamental L-tetrahedron τ_0 with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and (p,q,r), where p, q, and r are integers such that $0 \leqslant p < r$, $0 \leqslant q < r$, and $(p,r) = (q,r) = (p+q-1, r) = 1$.

In view of what we have said earlier it is now only necessary, in order to establish Theorem II, for us to verify that (9) is satisfied by the fundamental L-tetrahedron τ_0.† Now the volume $V(\tau_0)$ of τ_0 is $r/6$ and so,

† It might seem at first sight that we could now give a more explicit definition of τ_0. In fact, however, this does not appear to be so easy. For instance, one might be tempted at first to guess that neither p nor q could be greater than unity, however, the values 2, 5, and 7 respectively for p, q, and r give an example of a fundamental L-tetrahedron for which this supposition is false.

substituting this in (9), we see that we have only to show that

$$6\Psi_n^r(\tau_0) = (n-1)\{rn^2+(r-6)n+6\}. \tag{10}$$

In order to establish the formula (10) we note first that by homogeneity $\Psi_n^r(\tau_0) = \Psi_1^r(\tau_n)$ where τ_n is the L-tetrahedron with vertices

$$(0,0,0),\ (n,0,0),\ (0,n,0),\ \text{and}\ (np,nq,nr).$$

We count the number of points of L in the interior of τ_n by determining separately the number of such points (u,v,w) in the different planes with constant values for w. For this purpose we need the following lemma.

LEMMA III. *Let x, y be real numbers and let s be positive. Then the integer*

$$l = -2-[x]-[y]-[-x-y-s],$$

where $[x]$ denotes the greatest integer not exceeding x, is not less than -1 and the number of points (u,v) with integral coordinates in the interior of the triangle with vertices (x,y), $(x+s,y)$, and $(x,y+s)$ is $\frac{1}{2}l(l+1)$.

Proof. As s is positive while $x-[x]$ and $y-[y]$ are non-negative, we have

$$[-x+[x]-y+[y]-s] \leqslant -1,$$

so that

$$\begin{aligned} l &= -2-[x]-[y]-[-x-y-s] \\ &= -2-[-x+[x]-y+[y]-s] \\ &\geqslant -2-(-1) = -1. \end{aligned}$$

The points (u,v) in the interior of the triangle are the points satisfying

$$x < u,$$
$$y < v,$$

and

$$-x-y-s < -u-v.$$

But there are no points (u,v) with integral coordinates satisfying

$$x < u < [x]+1,$$

nor satisfying

$$y < v < [y]+1,$$

nor satisfying

$$-x-y-s < -u-v < [-x-y-s]+1.$$

Thus the number of lattice points in the interior of the triangle is the number of pairs (u,v) of integers satisfying

$$[x]+1 \leqslant u,$$
$$[y]+1 \leqslant v,$$

and

$$[-x-y-s]+1 \leqslant -u-v.$$

Writing

$$u-[x]-1 = u', \qquad v-[y]-1 = v',$$

we see that this is the number of pairs of integers (u', v') satisfying

$$u' \geqslant 0, \qquad v' \geqslant 0,$$

and
$$u'+v' \leqslant -3-[x]-[y]-[-x-y-s],$$

i.e.
$$u' \geqslant 0, \quad v' \geqslant 0, \quad \text{and} \quad u'+v' \leqslant l-1.$$

Since $l \geqslant -1$, this number is $\frac{1}{2}l(l+1)$ as required.

For any integer k, with $0 < k < nr$, the plane $z = k$ meets the tetrahedron τ_n in the triangle with vertices

$$\left(\frac{k}{r}p, \frac{k}{r}q, k\right), \quad \left(\frac{k}{r}p+\frac{nr-k}{r}, \frac{k}{r}q, k\right),$$

and
$$\left(\frac{k}{r}p, \frac{k}{r}q+\frac{nr-k}{r}, k\right).$$

So by the lemma the number of points of L in the interior of τ_n lying on the plane $z = k$ is $\frac{1}{2}l_k(l_k+1)$ where

$$\begin{aligned} l_k &= -2-\left[\frac{k}{r}p\right]-\left[\frac{k}{r}q\right]-\left[-\frac{k}{r}p-\frac{k}{r}q-\frac{nr-k}{r}\right] \\ &= n-2-\left[\frac{k}{r}p\right]-\left[\frac{k}{r}q\right]-\left[-\frac{k}{r}(p+q-1)\right]. \end{aligned}$$

Note also that we have $l_k \geqslant -1$ by the lemma.

In the special case when k is of the form rs where $0 < s < n$ we see that l_k reduces to
$$n-2-sp-sq+s(p+q-1) = n-s-2.$$

So the total number of points of L in τ_n on these planes $z = rs$ is

$$\begin{aligned} \sum_{s=1}^{n-1} \tfrac{1}{2}l_{rs}(l_{rs}+1) &= \sum_{s=1}^{n-1} \tfrac{1}{2}(n-s-2)(n-s-1) \\ &= \sum_{t=1}^{n-1} \tfrac{1}{2}(t-2)(t-1) = \tfrac{1}{6}(n-1)(n-2)(n-3). \end{aligned}$$

Now consider the case when $k = rs+t$ where $0 \leqslant s \leqslant n-1$ and $0 < t < r$. We have

$$\begin{aligned} l_k &= n-2-\left[sp+\frac{t}{r}p\right]-\left[sq+\frac{t}{r}q\right]-\left[-s(p+q-1)-\frac{t}{r}(p+q-1)\right] \\ &= n-s-2-\left[\frac{t}{r}p\right]-\left[\frac{t}{r}q\right]-\left[-\frac{t}{r}(p+q-1)\right]. \end{aligned}$$

But

$$\left[\frac{t}{r}p\right]+\left[\frac{t}{r}q\right]+\left[-\frac{t}{r}(p+q-1)\right] = \left[-\frac{t}{r}p+\left[\frac{t}{r}p\right]-\frac{t}{r}q+\left[\frac{t}{r}q\right]+\frac{t}{r}\right]$$

is of the form
$$[x+y+z]$$

where $\quad -1 < x \leqslant 0, \quad -1 < y \leqslant 0, \quad 0 < z < 1,$

and so takes one of the values -2, -1, or 0. We determine the precise value by use of our assumption that τ_0 is a fundamental L-tetrahedron and so contains no point of L other than the vertices. By the special cases of the above result with

$$n = 1 \quad \text{and} \quad k = t$$

and with

$$n = 1 \quad \text{and} \quad k = r-t$$

the numbers of points of L in τ_0 on the planes $z = t$ and $z = r-t$ are $\frac{1}{2}l(l+1)$ and $\frac{1}{2}l'(l'+1)$ respectively, where

$$l = -1-\left[\frac{t}{r}p\right]-\left[\frac{t}{r}q\right]-\left[-\frac{t}{r}(p+q-1)\right],$$

and

$$l' = -1-\left[\frac{r-t}{r}p\right]-\left[\frac{r-t}{r}q\right]-\left[-\frac{r-t}{r}(p+q-1)\right].$$

Since there are no such points we must have

$$l = 0 \text{ or } -1,$$

and

$$l' = 0 \text{ or } -1.$$

But as $(p,r) = 1$, $(q,r) = 1$, and $(p+q-1,r) = 1$, while $0 < t < r$, none of the ratios

$$\frac{t}{r}p,\ \frac{t}{r}q \text{ or } \frac{t}{r}(p+q-1)$$

can be integers. Hence

$$l+l' = -2-\left[\frac{t}{r}p\right]-\left[p-\frac{t}{r}p\right]-\left[\frac{t}{r}q\right]-\left[q-\frac{t}{r}q\right]-$$
$$-\left[-\frac{t}{r}(p+q-1)\right]-\left[-(p+q-1)+\frac{t}{r}(p+q-1)\right]$$
$$= -2-p+1-q+1+(p+q-1)+1 = 0.$$

Consequently we must have $l = l' = 0$. Thus

$$\left[\frac{t}{r}p\right]+\left[\frac{t}{r}q\right]+\left[-\frac{t}{r}(p+q-1)\right] = -1$$

and

$$l_k = n-s-1 \quad \text{when } k = rs+t.$$

So the number of points of L in the interior of τ_n lying on the planes of the form $z = rs+t$ where $0 \leqslant s \leqslant n-1$ and $0 < t < r$ is

$$\sum_{s=0}^{n-1}\sum_{t=1}^{r-1}\tfrac{1}{2}l_{rs+t}(l_{rs+t}+1) = (r-1)\sum_{s=0}^{n-1}\tfrac{1}{2}(n-s-1)(n-s)$$
$$= (r-1)\sum_{t=1}^{n}\tfrac{1}{2}t(t-1) = \tfrac{1}{6}(r-1)(n+1)n(n-1).$$

Thus $\Psi_n(\tau_0) = \Psi_1(\tau_n) = \frac{1}{6}(n-1)\{(n-2)(n-3)+(r-1)n(n+1)\}$
$$= \tfrac{1}{6}(n-1)\{rn^2+(r-6)n+6\}$$
as required.

It is perhaps worth remarking that if we retained the assumption that
$$(p,r) = 1, \quad (q,r) = 1, \quad \text{and} \quad (p+q-1,r) = 1,$$
but replaced the assumption that there is no point of L in the interior of τ_0, by the assumption that h points of L lie in the interior of τ_0, then a refinement of the above argument would lead to the formula
$$\Psi_n(\tau_0) = \tfrac{1}{6}(n-1)\{rn^2+(r-6)n+6\}+hn;$$
a result which is clearly consistent with the formula (4).

6. Concluding remarks

We conclude this paper with a brief consideration of the possibility of extending our results to polyhedra in space of dimension greater than three.

Let us suppose for a moment that in proving Theorem II we had started with the assumption that the volume $V(\Pi)$ of an L-polyhedron Π could be determined by a relation of the form
$$V(\Pi) = aL_n(\Pi)+bL(\Pi)+cL_n(\bar{\Pi})+dL(\bar{\Pi})+fN(\Pi)+gN(\bar{\Pi}),$$
and that we had set out to find the values of the constants a, b, c, d, f, and g. We should have been led, by considering the additive property of V, to the conclusion that
$$\begin{bmatrix}\{(a+2c)L_n(\pi)+(b+2d)L(\pi)+(f+2g)N(\pi)\} \\ -\{cL_n(\pi^*)+dL(\pi^*)+gN(\pi^*)\}\end{bmatrix} = 0,$$
π and π^* having here the same significance as they did previously. An obvious way in which to satisfy this condition would be in the first place to impose the conditions
$$a+2c = b+2d = f+2g = 0,$$
and then to determine the ratios of the constants c, d, and g so that the second bracket in the above expression vanished; the remaining constant of proportionality could then be determined by using the condition that the function V as defined above actually gives the volume of some particular polyhedron, e.g. some fundamental L-tetrahedron. Of course, there is no evidence on the face of it that this procedure will in fact lead us to a formula of the type for which we are looking, but in the case in question of L-polyhedra in three dimensions it does in fact do so. Now the success of this method depends, amongst other things, upon the existence of a set of ratios of the constants c, d, and g such that the expression
$$cL_n(\pi^*)+dL(\pi^*)+gN(\pi^*)$$

vanishes for an arbitrary singular L-path π^*. These ratios exist because $L_n(\pi^*)$ and $L(\pi^*)$ may be used in effect to count the respective numbers of 0- and 1-simplexes in a covering of π^* (whose Euler–Poincaré characteristic is $N(\pi^*)$) by a simplicial complex whose 1-simplexes are all L-segments containing no points of L in their interiors.

Suppose now that we are dealing with a lattice polyhedron in some space of dimension k greater than three. We shall be led, if we carry out an investigation analogous to that just described, to look for a means of counting the numbers of simplexes of various dimension in a simplicial complex of dimension $k-2$. The obvious way in which to do this will be to introduce not a single additional lattice L_n but a number of distinct such lattices. Thus in the case of a four-dimensional polyhedron we might hope to find a formula for the volume involving just two additional lattices; for, amongst other things, this would involve obtaining an expression for the Euler–Poincaré characteristic of a two-dimensional simplicial complex K in terms of the numbers of points common to K and each of these lattices, and an example of such a relation is the following

$$N(K) = -3L(K)+3L_2(K)-L_3(K). \tag{11}$$

[*Added in proof*, 11.3.57. If one carries through the procedure outlined above in this case, determining the final constant of proportionality by making the resulting formula valid for the fundamental parallelepiped of the integer lattice L, one obtains, as a conjectured formula for the volume $V(\Pi)$ of the polyhedron Π in four dimensions, the equation

$$72V(\Pi) = 2\{3L(\Pi)-3L_2(\Pi)+L_3(\Pi)+N(\Pi)\}- \\ -\{3L(\dot{\Pi})-3L_2(\dot{\Pi})+L_3(\dot{\Pi})+N(\dot{\Pi})\}.$$

A similar formula may be obtained using lattices L_m and L_n, m and n being any two distinct integers greater than unity, in place of L_2 and L_3, the success of the procedure up to this point not depending upon the fact that 2 and 3 are mutually prime.]

Expressions similar to (11), involving three additional lattices, may be found for three-dimensional simplicial complexes. Thus it would seem that for the cases of lower dimension at least the task of proving the additive property of a function V derived in the way indicated above would be fairly straightforward, though we should still be far from finally establishing its general validity.

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The University
Reading